ON THE FORMULATION OF REFINED THEORIES OF PLATES AND SHELLS

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The attempt to refine theories of plates and shells was started in [1,2], and at the present time many papers are devoted to this problem. These papers usually use one of a number of assumptions. A survey of them is beyond the scope of this note. We mention only the papers in which a certain error is specified at the outset, for example, of the order of h^4/L^4 compared to unity ($2h$ is the thickness and L is the transverse dimension of the plate) and the differential equations corresponding to this accuracy are obtained [3-6]. Boundary conditions to within this error were obtained in [6].

In the author's view [7-8] are of the greatest interest in regard to the methodology of formulating a theory, even though they cannot lay claim to consistency in the question of refinement.

We present below a quite general method of formulating refined theories of plates and shells which goes back to Reissner's work [7] for its ideas. It is based on a generalized variational principle of the non-linear theory of elasticity [9].

1. In [9] it was shown that the following assertion holds (the linearized form of the relation proved in the indicated reference is used here).

Among all displacements, stresses, and strains only those actually occur that make the functional

$$J = \iiint\limits_V \mathbf{Q}\mathbf{u}\, dV + \iint\limits_{S_{(p)}} \mathbf{P}_{(s)}\, \mathbf{u}\, dS + \iint\limits_{S_{(u)}} \sigma^{ik}\mathbf{r}_k n_i\, (\mathbf{u}_{(s)} - \mathbf{u})\, dS - \iiint\limits_V \left\{W - \sigma^{ik}\left[\varepsilon_{ik} - \frac{1}{2}(\nabla_i u_k + \nabla_k u_i)\right]\right\} dV \qquad (1.1)$$

have a stationary value.

Here V is the volume of the body; $\mathbf{Q}$ is the body force vector referred to a unit volume; $\mathbf{u}$ is the displacement vector; W is the strain energy per unit volume; S is the boundary of the volume V, whereby $S_{(p)}$ is the portion of the surface on which the surface traction vector $\mathbf{P}_{(s)}$ is prescribed and $S_{(u)}$ is the portion on which the displacement vector $\mathbf{u}_{(s)}$ is given; σ^{ik} are the contravariant components of the stress tensor in the reference frame $x^i (i = 1, 2, 3)$ with coordinate vectors $\mathbf{r}_i$; ε_{ik} are the covariant components of the strain tensor; $\nabla_i(\ldots)$ is the symbol for covariant differentiation with respect to the metric g_{ik}, $g_{ik} = (\mathbf{r}_i \times \mathbf{r}_k)$; and $\mathbf{n}$ is the unit interior normal vector to the surface S

$$u_k = \mathbf{u} \cdot \mathbf{r}_k, \qquad n_i = \mathbf{n} \cdot \mathbf{r}_i$$

If the material follows a linear elastic law then

$$2W = A^{ikmn} \varepsilon_{ik} \varepsilon_{mn} \tag{1.2}$$

where A^{ikmn} are the components of the tensor of elastic constants. For an isotropic body

$$A^{ikmn} = \frac{E}{(1+\mu)(1-2\mu)} (\mu g^{ik} g^{mn} + (1-2\mu) g^{im} g^{kn}) \tag{1.3}$$

Here E is the modulus of elasticity and μ is the coefficient of transverse contraction.

In the functional J the displacements $\mathbf{u}$, the stresses σ^{ik} and the strains ε_{ik} are allowed to vary independently. It is assumed that $\delta\mathbf{Q} = \delta\mathbf{P}_{(s)} = 0$.

If the plate or shell is symmetrically constructed with respect to some mean surface σ which is sufficiently smooth, and if the plate or shell has a bounding section Σ, whose generators are normal to σ, then the first variation of the functional J may be represented in the form

$$\begin{aligned}\delta J = &\iint_{\sigma} \int_{-h}^{h} (\nabla_i \sigma^{ik} + Q^k) \delta u_k \sqrt{\frac{g}{a}}\, dz\, d\sigma + \iint_{\sigma_+} (P_+{}^k + \sigma_+{}^{ik} n_i{}^+) \delta u_k{}^+ d\sigma_+ + \\ &+ \iint_{\sigma_-} (P_-{}^k + \sigma_-{}^{ik} n_i{}^-) \delta u_k{}^- d\sigma_- + \iint_{\Sigma_{(p)}} (P_{(\Sigma)}^k + \sigma^{ik} \nu_i) \delta u_k\, d\Sigma_{(p)} + \\ &+ \iint_{\Sigma_{(u)}} (u_k^{(\Sigma)} - u_k) \delta \sigma^{ik} \nu_i\, d\Sigma_{(u)} + \iint_{\sigma} \int_{-h}^{h} (\sigma^{ik} - A^{ikmn} \varepsilon_{mn}) \delta \varepsilon_{ik} \sqrt{\frac{g}{a}}\, dz\, d\sigma + \\ &+ \iint_{\sigma} \int_{-h}^{h} \left[\varepsilon_{ik} - \frac{1}{2} (\nabla_i u_k + \nabla_k u_i)\right] \delta \sigma^{ik} \sqrt{\frac{g}{a}}\, dz\, d\sigma = 0\end{aligned} \tag{1.4}$$

$$(g = \det \| g_{ik} \|)$$

Here $2h$ is the thickness of the plate or shell, which in general is a function of points along the mean surface σ; $x^3 = z$ is a coordinate reckoned along the normal to the mean surface; the signs (+) or (−) attached to a quantity indicate that the quantity is calculated either at $z = h$ or $z = -h$; the surfaces $z = h$ and $z = -h$ are denoted by σ_+ and σ_-, respectively; $\Sigma_{(p)}$ is the portion of the surface Σ on which external loads are specified, $\Sigma_{(u)}$ is the portion on which displacements are specified; $\mathbf{v}$ is the unit interior normal to the surface Σ. The line of intersection of the surfaces σ and Σ we denote by C. Further, we shall assume that on the surface σ there is a coordinate net $x^\alpha (\alpha = 1, 2)$, with coordinate vectors $\rho_\alpha = \mathbf{r}_\alpha|_{z=0}$. The positive sense of the coordinate z will be along the direction of $\mathbf{m}$, determined from the relationship

$$\mathbf{m}c_{\alpha\beta} = \rho_\alpha \times \rho_\beta$$

Here $c_{\alpha\beta}$ is the discriminant tensor on the surface σ, the nonvanishing components of which are

$$c_{12} = -c_{21} = \sqrt{a}, \qquad a = \det \| a_{\alpha\beta} \|, \qquad a_{\alpha\beta} = \rho_\alpha \cdot \rho_\beta$$

Here and in the sequel Greek indices of tensor character take on the values 1 and 2, while Latin indices of the same character take on the values 1, 2, 3. Remaining indices are enclosed in parentheses.

We specify the displacements, stresses, and strains in the form

$$\begin{gathered} u_\alpha = u_\alpha^{(0)} + zu_\alpha^{(1)} + z^2 u_\alpha^{(0)} + z^3 u_\alpha^{(3)}, \qquad w = u_3 = w_{(0)} + zw_{(1)} + z^2 w_{(2)} \\ \sigma^{\alpha\beta} = \sigma^{\alpha\beta}_{(0)} + z\sigma^{\alpha\beta}_{(1)} + z^2\sigma^{\alpha\beta}_{(2)} + z^3\sigma^{\alpha\beta}_{(3)}, \qquad \sigma^{\alpha 3} = \sigma^{\alpha 3}_{(0)} + z\sigma^{\alpha 3}_{(1)} + z^2\sigma^{\alpha 3}_{(2)} \\ \sigma^{33} = \sigma^{33}_{(0)} + z\sigma^{33}_{(1)}, \qquad \varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta(0)} + z\varepsilon_{\alpha\beta(1)} + z^2\varepsilon_{\alpha\beta(2)} + z^3\varepsilon_{\alpha\beta(3)} \\ \varepsilon_{\alpha 3} = \varepsilon_{\alpha 3(0)} + z\varepsilon_{\alpha 3(1)} + z^2\varepsilon_{\alpha 3(2)}, \qquad \varepsilon_{33} = \varepsilon_{33(0)} + z\varepsilon_{33(1)} + z^2\varepsilon_{33(2)} + z^3\varepsilon_{33(3)} \end{gathered} \tag{1.5}$$

i.e. in functions of the coordinate z we approximate the displacements, stresses, and strains by mean forms, while the functions $u_\alpha^{(i)}$, $w_{(i)}$, $\sigma^{mn}_{(i)}$, $\varepsilon_{mn(i)}$, which depend on the coordinates x^α, we determine from Equations (1.4). As will be seen subsequently, the arbitrariness introduced by the relations (1.5) reduces to the arbitrariness corresponding to the specification of the law of variation of the displacements across the thickness.

2. Limiting the investigation to plates, we obtain from Equation (1.4):

Equations of equilibrium

$$\nabla_\alpha \frac{2h^{n+1}}{n+1}\sigma^{\alpha\beta}_{(0)} + \nabla_\alpha \frac{2h^{n+3}}{n+3}\sigma^{\alpha\beta}_{(2)} - n\sigma^{3\beta}_{(1)}\frac{2h^3}{3} + h^n(P^\beta_+ + P^\beta_-) + \int_{-h}^{h} Q^\beta z^n\,dz = 0$$

$$\nabla_\alpha \frac{2h^{n+3}}{n+3}\sigma^{\alpha\beta}_{(1)} + \nabla_\alpha \frac{2h^{n+5}}{n+5}\sigma^{\alpha\beta}_{(3)} - (n+1)\sigma^{3\beta}_{(0)}\frac{2h^{n+1}}{n+1} - (n+1)\sigma^{3\beta}_{(2)}\frac{2h^{n+3}}{n+3} + $$

$$+ h^{n+1}(P^\beta_+ - P^\beta_-) + \int_{-h}^{h} Q^\beta z^{n+1}\,dz = 0 \qquad (2.1)$$

$$\nabla_\alpha \frac{2h^{n+1}}{n+1}\sigma^{\alpha 3}_{(0)} + \nabla_\alpha \frac{2h^{n+3}}{n+3}\sigma^{\alpha 3}_{(2)} - n\sigma^{33}_{(1)}\frac{2h^3}{3} + h^n(P^3_+ + P^3_-) + \int_{-h}^{h} Q^3 z^n\,dz = 0$$

$$\nabla_\alpha \frac{2h^3}{3}\sigma^{\alpha 3}_{(1)} - \sigma^{33}_{(0)}2h + h(P^3_+ - P^3_-) + \int_{-h}^{h} Q^3 z\,dz = 0, \qquad n = 0, 2$$

Elasticity relationship

$$\sigma^{ik}_{(j)} = A^{ikmn}\varepsilon_{mn(j)} \qquad (2.2)$$

Strain-displacement dependence

$$2\varepsilon_{\alpha\beta(i)} = \nabla_\alpha u_{\beta(i)} + \nabla_\beta u_{\alpha(i)}, \qquad 2\varepsilon_{\alpha 3(j)} = \nabla_\alpha w_{(j)} + (j+1)\,u_{\alpha\,(j+1)} \qquad (2.3)$$

$$\varepsilon_{33\,(0)} = w_{(1)}, \quad \varepsilon_{33\,(1)} = 2w_{(2)}, \quad \varepsilon_{33\,(2)} = -\frac{\mu}{1-\mu}\nabla_\alpha u^\alpha_{(2)}, \quad \varepsilon_{33\,(3)} = -\frac{\mu}{1-\mu}\nabla_\alpha u^\alpha_{(3)}$$

Intrinsic static conditions

$$\left\{\sigma^{\alpha\beta}_{(0)}\frac{2h^{n+1}}{n+1} + \sigma^{\alpha\beta}_{(2)}\frac{2h^{n+3}}{n+3}\right\}\nu_\alpha\nu_\beta + \int_{-h}^{h} P^\alpha_{(\Sigma)}\nu_\alpha z^n\,dz = 0$$

$$\left\{\sigma^{\alpha\beta}_{(0)}\frac{2h^{n+1}}{n+1} + \sigma^{\alpha\beta}_{(2)}\frac{2h^{n+3}}{n+3}\right\}\nu_\alpha\tau_\beta + \int_{-h}^{h} P^\alpha_{(\Sigma)}\tau_\alpha z^n\,dz = 0$$

$$\left\{\sigma^{\alpha\beta}_{(1)}\frac{2h^{n+3}}{n+3} + \sigma^{\alpha\beta}_{(3)}\frac{2h^{n+5}}{n+5}\right\}\nu_\alpha\nu_\beta + \int_{-h}^{h} P^\alpha_{(\Sigma)}\nu_\alpha z^{n+1}\,dz = 0$$

$$\left\{\sigma^{\alpha\beta}_{(1)}\frac{2h^{n+3}}{n+3} + \sigma^{\alpha\beta}_{(3)}\frac{2h^{n+5}}{n+5}\right\}\nu_\alpha\tau_\beta + \int_{-h}^{h} P^\alpha_{(\Sigma)}\tau_\alpha z^{n+1}\,dz = 0 \qquad (2.4)$$

$$\left\{\sigma^{\alpha 3}_{(0)}\frac{2h^{n+1}}{n+1} + \sigma^{\alpha 3}_{(2)}\frac{2h^{n+3}}{n+3}\right\}\nu_\alpha + \int_{-h}^{h} P^3_{(\Sigma)} z^n\,dz = 0$$

$$\sigma^{\alpha 3}_{(1)}\frac{2h^3}{3}\nu_\alpha + \int_{-h}^{h} P^3_{(\Sigma)} z\,dz = 0 \qquad (n = 0, 2)$$

Intrinsic geometric boundary conditions

$$\left\{u^{\alpha}_{(0)} \frac{2h^{n+1}}{n+1} + u^{\alpha}_{(2)} \frac{2h^{n+3}}{n+3}\right\} \nu_\alpha = \int_{-h}^{h} u^{\alpha}_{(\Sigma)} \nu_\alpha z^n \, dz$$

$$\left\{u^{\alpha}_{(0)} \frac{2h^{n+1}}{n+1} + u^{\alpha}_{(2)} \frac{2h^{n+3}}{n+3}\right\} \tau_\alpha = \int_{-h}^{h} u^{\alpha}_{(\Sigma)} \tau_\alpha z^n \, dz$$

$$\left\{u^{\alpha}_{(1)} \frac{2h^{n+3}}{n+3} + u^{\alpha}_{(3)} \frac{2h^{n+5}}{n+5}\right\} \nu_\alpha = \int_{-h}^{h} u^{\alpha}_{(\Sigma)} \nu_\alpha z^{n+1} \, dz$$

$$\left\{u^{\alpha}_{(1)} \frac{2h^{n+3}}{n+3} + u^{\alpha}_{(3)} \frac{2h^{n+5}}{n+5}\right\} \tau_\alpha = \int_{-h}^{h} u^{\alpha}_{(\Sigma)} \tau_\alpha z^{n+1} \, dz$$

$$w_{(0)} \frac{2h^{n+1}}{n+1} + w_{(2)} \frac{2h^{n+3}}{n+3} = \int_{-h}^{h} w_{(\Sigma)} z^n \, dz \tag{2.5}$$

$$w_{(1)} \frac{2h^3}{3} = \int_{-h}^{h} w_{(\Sigma)} \, z dz \qquad (n = 0, 2)$$

Here $\mathbf{v} = \nu^\alpha \rho_\alpha$, $\mathbf{\tau} = \tau^\alpha \rho_\alpha$ are interior unit normals to the bounding surface Σ and tangents to the line C, oriented so that the triplet of vectors $\mathbf{\tau}$, $\mathbf{v}$, $\mathbf{n}$ forms a right-handed triad; $\nabla_\alpha(\ldots)$ is the symbol for covariant differentiation with respect to the metric $a_{\alpha\beta}$. In addition, we have introduced a simplification into Equation (2.1) that corresponds to neglecting, compared to unity, the squares of the derivatives of the thickness along the coordinates of the middle surface. Since the plate is symmetrically constructed, the problem of determining its state of stress decomposes into a bending problem and a problem of determining the contraction of the thickness. As is easily seen, the order of the differential equations (2.1) corresponds to the number of boundary conditions (2.4) or (2.5).

3. Consider a circular isotropic plate of radius r under symmetric bending. Assuming $x^1 = \eta$, $0 \leqslant \eta \leqslant 1$ ($\rho = \eta r$ is the distance from an arbitrary point to the axis of the plate) and assuming that the thickness varies only along the radius, we obtain

$$\varepsilon_{11\,(i)} = r^2 \frac{du^1_{(i)}}{d\eta}, \qquad \varepsilon_{22\,(i)} = \eta r^2 u^1_{(i)}, \qquad \varepsilon_{33\,(0)} = w_{(1)}, \qquad \varepsilon_{33\,(1)} = 2w_{(2)}$$

$$\varepsilon_{33\,(2)} = -\frac{\mu}{1-\mu} \frac{1}{\eta} \frac{d}{d\eta} \eta u^1_{(2)}, \qquad \varepsilon_{33\,(3)} = -\frac{\mu}{1-\mu} \frac{1}{\eta} \frac{d}{d\eta} \eta u^1_{(3)}$$

$$2\varepsilon_{13\,(i)} = \frac{dw_{(i)}}{d\eta} + (i+1)\, r^2 u^1_{(i+1)} \tag{3.1}$$

$$\sigma^{11}_{(i)} = \frac{E}{(1+\mu)(1-2\mu)}\frac{1}{r^2}\left\{\frac{\mu}{\eta}u^1_{(i)} + (1-\mu)\frac{du^1_{(i)}}{d\eta}\right\} + \frac{E\mu}{(1+\mu)(1-2\mu)}\frac{1}{r^2}\varepsilon_{33\,(i)}$$

$$\sigma^{22}_{(i)} = \frac{E}{(1+\mu)(1-2\mu)}\frac{1}{r^2\eta^2}\left\{\mu\frac{du^1_{(i)}}{d\eta} + (1-\mu)\frac{u^1_{(i)}}{\eta}\right\} + \frac{E\mu}{(1+\mu)(1-2\mu)}\frac{1}{r^2\eta^2}\varepsilon_{33\,(i)} \quad (3.2)$$

$$\sigma^{13}_{(i)} = \frac{E}{2(1+\mu)r^2}\left\{\frac{dw_{(i)}}{d\eta} + (i+1)r^2u^1_{(i+1)}\right\}$$

$$\sigma^{33}_{(i)} = \frac{E(1-\mu)(i+1)}{(1+\mu)(1-2\mu)}w_{(i+1)} + \frac{E\mu}{(1+\mu)(1-2\mu)}\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(i)}$$

The equations of bending take on the form

$$\frac{2h^{n+3}}{n+3}\frac{1}{r^2}\left\{(1-\mu)\frac{d}{d\eta}\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(1)} + 2\mu\frac{dw_{(2)}}{d\eta}\right\} + \quad (3.3)$$

$$+\frac{2h^{n+2}}{r^2}\frac{dh}{d\eta}\left\{\frac{\mu}{\eta}u^1_{(1)} + (1-\mu)\frac{du^1_{(1)}}{d\eta} + 2\mu w_{(2)}\right\} + \frac{2h^{n+5}}{n+5}\frac{1-2\mu}{(1-\mu)r^2}\frac{d}{d\eta}\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(3)} +$$

$$+\frac{2h^{n+4}(1-2\mu)}{r^2(1-\mu)}\frac{dh}{d\eta}\left\{\frac{\mu}{\eta}u^1_{(3)} + \frac{du^1_{(3)}}{d\eta}\right\} - \frac{(1-2\mu)h^{n+1}}{r^2}\left\{\frac{dw_0}{d\eta} + r^2u^1_{(1)}\right\} -$$

$$-\frac{(n+1)(1-2\mu)h^{n+3}}{(n+3)r^2}\left\{\frac{dw_{(2)}}{d\eta} + 3r^2u^1_{(3)}\right\} +$$

$$+\frac{(1+\mu)(1-2\mu)}{E}\left\{h^{n+1}(P^1_+ - P^1_-) + \int_{-h}^{h} Q^1z^{n+1}\,dz\right\} = 0$$

$$\frac{2h^{n+1}}{(n+1)r^2}\left\{\frac{1}{\eta}\frac{d}{d\eta}\eta\frac{dw_0}{d\eta} + r^2\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(1)}\right\} + \frac{2h^n}{r^2}\frac{dh}{d\eta}\left\{\frac{dw_0}{d\eta} + r^2u^1_{(1)}\right\} + \quad (3.4)$$

$$+\frac{2h^{n+3}}{(n+3)r^2}\left\{\frac{1}{\eta}\frac{d}{d\eta}\eta\frac{dw_{(2)}}{d\eta} + r^2\frac{1}{\eta}\frac{d}{d\eta}\eta u^{(1)}_{(3)}\right\} + \frac{2h^{n+2}}{r^2}\frac{dh}{d\eta}\left\{\frac{dw_{(2)}}{d\eta} + r^2u^1_{(3)}\right\} -$$

$$-\frac{4nh^3}{3(1-2\mu)}\left\{2(1-\mu)w_{(2)} + \mu\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(1)}\right\} +$$

$$+\frac{2(1+\mu)}{E}\left\{h^n(P_+^3 + P_-^3) + \int_{-h}^{h} Q^3z^n\,dz\right\} = 0$$

The equations of contraction of the plate take on the form

$$\frac{2h^{n+1}}{(n+1)r^2}\left\{(1-\mu)\frac{d}{d\eta}\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(0)} + \mu\frac{dw_{(1)}}{d\eta}\right\} + \frac{2h^n}{r^2}\frac{dh}{d\eta}\left\{\frac{\mu}{\eta}u^1_{(0)} + (1-\mu)\frac{du^1_{(0)}}{d\eta} + \mu w_{(1)}\right\} +$$

$$+\frac{2h^{n+3}}{n+3}\frac{1-2\mu}{(1-\mu)r^2}\frac{d}{d\eta}\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(2)} + \frac{2h^{n+2}}{r^2}\frac{1-2\mu}{1-\mu}\frac{dh}{d\eta}\left\{\frac{\mu}{\eta}u^1_{(2)} + \frac{du^1_{(2)}}{d\eta}\right\} - \quad (3.5)$$

$$-\frac{nh^3(1-2\mu)}{3r^2}\left\{\frac{dw_{(1)}}{d\eta} + 2r^2u^1_{(2)}\right\} + \frac{(1+\mu)(1-2\mu)}{E}\left\{h^n(P_+^1 + P_-^1) + \int_{-h}^{h} Q^1z^n\,dz\right\} = 0$$

$$\frac{2h^3}{3r^2}\left\{\frac{1}{\eta}\frac{d}{d\eta}\eta\frac{dw_{(1)}}{d\eta} + r^2\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(2)}\right\} + \frac{2h^2}{r^2}\frac{dh}{d\eta}\left\{\frac{dw_{(1)}}{d\eta} + r^2u^1_{(2)}\right\} - \quad (3.6)$$

$$-\frac{4h}{1-2\mu}\left\{(1-\mu)w_{(1)} + \mu\frac{1}{\eta}\frac{d}{d\eta}\eta u^1_{(1)}\right\} + \frac{2(1+\mu)}{E}\left\{h(P_+^3 - P_-^3) + \int_{-h}^{h} Q_z^3\,dz\right\} = 0$$

The method of formulating refined theories of plates and shells that has been presented is characterized by a transparency that allows one to obtain the equations of the problem without unnecessary contrivances. However, it must be remarked that Equations (2.1) or their simpler version Equations (3.3) to (3.6) can be solved only by means of modern computers. For this the method of finite differences may be used. In addition, the variational equations (1.4) allow the possibility of finding the state of stress in plates and shells in which boundary conditions that vary with thickness are specified on the bounding cross sections. For example, a thick circular plate may be rigidly clamped on the segment $-h \leqslant z \leqslant \zeta$ of the bounding cross section and free of loads on the segment $\zeta \leqslant z \leqslant h$ $(-h < \zeta < h)$.

The question of the corrections that are introduced at the expense of refinements in the equations of equilibrium and boundary conditions is discussed in [6].

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